PROOF OF EULER'S RELATION

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Eulers Relation can be derived from **Taylors Series**, defined as:

$$f(x) = \sum_{k=0}^{\infty}\left(\frac{f^{(k)}(0)}{k!}\cdot x^k\right)$$

Brook Taylor 1685 - 1731

note: $f^{(k)}(0)$ represents the kth differential of the funtion f

also:

$$f(t) = \sin(t) \quad f(0) = 0$$

$$f^{i}(t) = \cos(t) \quad f'(0) = 1$$

$$f^{ii}(t) = -\sin(t) \quad f''(0) = 0$$

$$f^{iii}(t) = -\cos(t) \quad f'''(0) = -1$$

$$f^{iv}(t) = \sin(t) \quad f^{iv}(0) = 0$$

...

So:

(i) $$\sin(x) = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{(2k+1)!}\right)\cdot x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

and

(ii) $$\cos(x) = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{(2k)!}\right)\cdot x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Notice that sin(x) has all odd denominators, while cos(x) has all even denominators. The series can be combined to create something that looks similar to Taylors Series of e^x:

(iii) $$e^x = \sum_{k=0}^{\infty}\left(\frac{x^k}{k!}\right) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

e^x is the only mathmatical expression that is equal to it's own differential.

Definition of Eulers Relation:

$$e^{ix} = \cos(x) + \sin(x) \cdot i$$

Where $i = \sqrt{-1}$

Note the following equalities: (iv)

$$i^{-2} = (-1)^{-1} = -1$$

$$i^{-1} = (-1)^{-1/2} = -i$$

$$i^{0} = 1$$

$$i^{1} = \sqrt{-1} = i$$

$$i^{2} = \left(\sqrt{-1}\right)^{2} = -1$$

$$i^{3} = i^{2} \cdot i = -i$$

$$i^{4} = i^{2} \cdot i^{2} = 1$$

$$i^{5} = i^{3} \cdot i^{2} = i$$

$$i^{6} = \left(i^{2}\right)^{3} = -1$$

$$\ldots$$

$$\ldots \quad -1 \quad -i \quad 1 \quad i \quad \ldots$$

Interestingly:

$$\sum_{k=0}^{\infty}\left(i^{4k}\right) = 0$$

and for all positive integer values of n:

$$\sum_{k=-4n}^{4n}\left(i^{k}\right) = 0$$

PROOF

Substitute ix into the true equation shown in (iii):

$$e^{ix} = \sum_{k=0}^{\infty}\left(\frac{(ix)^k}{k!}\right) = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!}\ldots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\cdot\frac{x^3}{3!} + \frac{x^4}{4!} + i\cdot\frac{x^5}{5!}\ldots$$ *see* (iv)

Factor:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots\right)\cdot i$$

Replace first term with equation (ii) and second term with equation (i):

$$e^{ix} = \cos(x) + \sin(x)\cdot i$$ (v)

Similarly:

$$e^{-ix} = \cos(x) - \sin(x)\cdot i$$ (vi)

Add (v) and (vi) to eliminate the $\sin(x)\cdot i$ terms

$$e^{ix} + e^{-ix} = 2\cdot\cos(x)$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \text{ (vii)}$$

or subtract (v) from (vi) to eliminate the $\cos(x)$ terms:

$$e^{ix} - e^{-ix} = 2i\cdot\sin(x)$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \text{ (viii)}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\tan(x) = \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}\cdot i$$

These equations can be substituted back into Eulers Relation (v) showing internal consistency:

$$e^{ix} = \left(\frac{e^{ix} + e^{-ix}}{2}\right) + \left(\frac{e^{ix} - e^{-ix}}{2i}\right)\cdot i$$ i's cancel

$$e^{ix} = \frac{e^{ix} + e^{-ix} + e^{ix} - e^{-ix}}{2}$$ e^{-ix} and $-e^{-ix}$ cancel

$$e^{ix} = \frac{2\cdot e^{ix}}{2}$$

Proof of: $i^i = e^{-\frac{\pi}{2}}$

Substitute x with $\frac{\pi}{2}$ in Eulers Relation (v)

$$e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\cdot i$$

$$e^{i\frac{\pi}{2}} = 0 + 1\cdot i = i$$

$$\frac{-1}{i} = \frac{-1}{\sqrt{-1}}\cdot\frac{\sqrt{-1}}{\sqrt{-1}} = -1\cdot\frac{\sqrt{-1}}{-1} = \sqrt{-1} = i$$

$$e^{\frac{\pi}{2i}} = i$$

$$\left[\left(e^{-\frac{\pi}{2}}\right)^{\frac{1}{i}}\right]^{i} = i^i$$

$$\therefore i^i = e^{-\frac{\pi}{2}} \approx 0.20787$$

Rearranging:

$$\pi = 2\cdot\log_e\left(i^{-1}\right) \approx 3.1415926$$

$$e = i^{-\frac{2i}{\pi}} = \sum_{k=0}^{\infty}\left(\frac{1}{k!}\right) \approx 2.718281$$

see (iii)